

九十五學年第二學期 PHYS2320 電磁學 期中考 II 試題(共兩頁)

[Griffiths Ch. 9] 2007/05/17, 10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

Boundary conditions: $D_1^\perp - D_2^\perp = \sigma_f$, $\mathbf{E}_1'' - \mathbf{E}_2'' = 0$, $B_1^\perp - B_2^\perp = 0$, $\mathbf{H}_1'' - \mathbf{H}_2'' = (\mathbf{K}_f \times \hat{\mathbf{n}})$.

The wave equation: $\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$

1. (12%, 8%)

(a) Which functions satisfy the wave equations? Check by explicit differentiation.

$$f_1(z, t) = Ae^{-b(z-vt)^2}, \quad f_2(z, t) = \frac{A}{b(z-vt)^2 + 1},$$

$$f_3(z, t) = Ae^{-b(z^2+vt)}, \quad f_4(z, t) = A \sin(bz) \cos^2(bvt).$$

(b) Show that $f_5(z, t) = A \cos(bz) \cos(bvt)$ satisfies the wave equation, and express it as the sum of two traveling waves.

2. (7%, 7%, 6%) The intensity ($I = \langle S \rangle$) of sunlight hitting the earth is about 1320 W/m^2 .

(a) Calculate the average energy density $\langle u \rangle$.

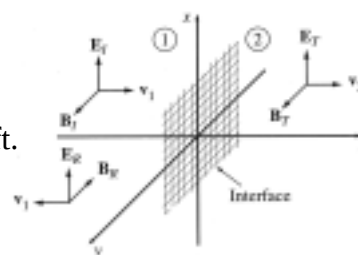
(b) Find the amplitude of the electric field E_0 .

(c) Find the force on a perfect reflector of area 1 m^2 if sunlight strikes normally on it.

[Hint: $\epsilon_0 = 8.8 \times 10^{-12} \text{ C}^2/(\text{N} \cdot \text{m}^2)$].

3. (7%, 7%, 6%) A plane wave approaches the interface from the left.

$$\text{Incident wave: } \begin{cases} \mathbf{E}_I(z, t) = E_{0I} \cos(k_1 z - \omega t) \hat{\mathbf{x}} \\ \mathbf{B}_I(z, t) = \frac{1}{v_1} E_{0I} \cos(k_1 z - \omega t) \hat{\mathbf{y}} \end{cases}$$



(a) Write down the reflected wave and the transmitted wave in terms of E_{0R} and E_{0T} , respectively.

(b) Write down the four boundary conditions, if there is no free charge and no free current at the interface.

(c) Find the reflection coefficient R and the transmission coefficient T

[Hint: Assume two media are linear.]

4. (12%, 8%) The skin depth is defined as $d \equiv \frac{1}{\kappa} = \frac{1}{\omega} \sqrt{\frac{2}{\epsilon\mu}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{-1/2}$

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $(2/\sigma)\sqrt{\epsilon/\mu}$ and in a good conductor $\sigma \gg \omega\epsilon$ is $\lambda/2\pi$.

(b) Find the skin depth of salt water ($\epsilon = 80\epsilon_0$, $\mu = \mu_0$, $\sigma = 22 (\Omega \cdot \text{m})^{-1}$) at 60 Hz and 60 GHz. Is it a poor conductor or a good conductor?

[Hint: $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$].

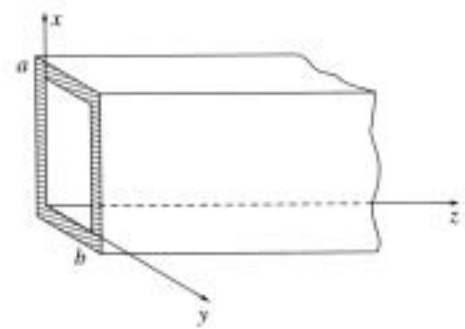
5. (10%, 10%) The field of the TE_{10} waveguide mode is

$$B_z = B_0 \cos(\pi x/a) e^{i(kz - \omega t)} \quad \text{and} \quad B_x = \frac{-ika}{\pi} B_0 \sin(\pi x/a) e^{i(kz - \omega t)}$$

$$E_y = \frac{i\omega\mu a}{\pi} B_0 \sin(\pi x/a) e^{i(kz - \omega t)}$$

(a) If $a = 7.2 \text{ cm}$ and $b = 3.4 \text{ cm}$, find the cutoff frequency ($f_{mn} = \omega_{mn}/2\pi$) of this mode.

(b) Consider the resonant cavity by closing off the two ends of the waveguide, at $z = 0$ and $z = d$, making a perfect conducting empty box. Using the new boundary condition, determine the electric and magnetic fields.



1.

(a)

$$f_1(z, t) = Ae^{-b(z-vt)^2} \dots (\circ)$$

$$\text{Let } x = z - vt, \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} = \frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 x}{\partial z^2} = \frac{\partial^2 f}{\partial x^2},$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} = -v \frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial z^2} = -v \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 x}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

$$f_2(z, t) = \frac{A}{b(z-vt)^2 + 1}, \dots (\circ)$$

$$\text{Let } x = z - vt, \text{ 同上 } \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$$f_3(z, t) = Ae^{-b(z^2+vt)}, \dots (\text{X})$$

$$\frac{\partial f}{\partial z} = -2Abze^{-b(z^2+vt)}, \quad \frac{\partial^2 f}{\partial z^2} = 4Ab^2 z^2 e^{-b(z^2+vt)} - 2Abe^{-b(z^2+vt)}$$

$$\frac{\partial f}{\partial t} = -Abve^{-b(z^2+vt)}, \quad \frac{\partial^2 f}{\partial t^2} = Ab^2 v^2 e^{-b(z^2+vt)}, \quad \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = Ab^2 e^{-b(z^2+vt)}$$

$$f_4(z, t) = A \sin(bz) \cos^2(bvt), \dots (\text{X})$$

$$\frac{\partial^2 f}{\partial z^2} = -b^2 A \sin(bz) \cos^2(bvt),$$

$$\frac{\partial f}{\partial t} = -2bAv \sin(bz) \cos(bvt) \sin(bvt), \quad \frac{\partial^2 f}{\partial t^2} = 2b^2 Av^2 \sin(bz) (\sin^2(bvt) - \cos^2(bvt))$$

$$\frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 2b^2 A \sin(bz) (\sin^2(bvt) - \cos^2(bvt))$$

(b)

$$\frac{\partial^2 f}{\partial z^2} = -Ab^2 \cos(bz) \cos(bvt), \quad \frac{\partial^2 f}{\partial t^2} = -Ab^2 v^2 \cos(bz) \cos(bvt), \quad \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = -Ab^2 \cos(bz) \cos(bvt)$$

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$$

$$f = A \cos(bz) \cos(bvt) = \frac{A}{2} (\cos(b(z+vt)) + \cos(b(z-vt)))$$

2.

$$(a) I = \langle S \rangle = c \langle u \rangle \rightarrow \langle u \rangle = 1320 / (3 \times 10^8) = 4.4 \times 10^{-6} \text{ J/m}^3.$$

$$(b) \langle u \rangle = \frac{1}{2} \epsilon_0 E_0^2 \rightarrow E_0 = \sqrt{(2 \times 4.4 \times 10^{-6}) / (8.8 \times 10^{-12})} = 1000 \text{ (V/m)}$$

$$(c) \frac{F}{A} = \frac{\Delta p}{A \Delta t} = \frac{2 \Delta U}{A c \Delta t} = \frac{2 S A}{A c} = \frac{2 S}{c} = 2u, \quad F = 8.8 \times 10^{-6} \text{ N}$$

3.

$$(a) \quad \text{Reflected wave: } \begin{cases} \mathbf{E}_R(z, t) = E_{0R} \cos(-k_1 z - \omega t) \hat{\mathbf{x}} \\ \mathbf{B}_R(z, t) = -\frac{1}{v_1} E_{0R} \cos(-k_1 z - \omega t) \hat{\mathbf{y}} \end{cases}$$

$$\text{Transmitted wave: } \begin{cases} \mathbf{E}_T(z, t) = E_{0T} \cos(k_2 z - \omega t) \hat{\mathbf{x}} \\ \mathbf{B}_T(z, t) = \frac{1}{v_2} E_{0T} \cos(k_2 z - \omega t) \hat{\mathbf{y}} \end{cases}$$

(b) For linear media, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

$$\begin{aligned} D_1^\perp - D_2^\perp = 0 \quad \mathbf{E}_1'' - \mathbf{E}_2'' = 0 \quad \epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = 0 \quad \mathbf{E}_1'' - \mathbf{E}_2'' = 0 \\ B_1^\perp - B_2^\perp = 0 \quad \mathbf{H}_1'' - \mathbf{H}_2'' = 0 \quad B_1^\perp - B_2^\perp = 0 \quad \frac{1}{\mu_1} \mathbf{B}_1'' - \frac{1}{\mu_2} \mathbf{B}_2'' = 0 \end{aligned} \Rightarrow$$

(c) Normal incident: no components perpendicular to the surface.

Normal boundary conditions: no normal components, so omit.

Tangential boundary conditions:

$$\begin{aligned} \mathbf{E}_1'' - \mathbf{E}_2'' = 0 \quad E_{0I} + E_{0R} = E_{0T} \\ \frac{1}{\mu_1} \mathbf{B}_1'' - \frac{1}{\mu_2} \mathbf{B}_2'' = 0 \Rightarrow \frac{1}{\mu_1 v_1} (\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2} \tilde{E}_{0T} \Rightarrow (\tilde{E}_{0I} - \tilde{E}_{0R}) = \beta \tilde{E}_{0T}, \text{ where } \beta = \frac{\mu_1 v_1}{\mu_2 v_2} \end{aligned}$$

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta} \right) \tilde{E}_{0I}, \text{ Reflection coefficient } R \equiv \frac{I_R}{I_I} = \left(\frac{1-\beta}{1+\beta} \right)^2$$

$$\tilde{E}_{0T} = \left(\frac{2}{1+\beta} \right) \tilde{E}_{0I}, \text{ Transmission coefficient } T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left(\frac{2}{1+\beta} \right)^2$$

4.

$$(a) \quad d \equiv \frac{1}{\kappa} = \frac{1}{\omega} \sqrt{\frac{2}{\epsilon \mu} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2} - 1 \right]}^{-1/2}$$

poor conductor:

$$\frac{\sigma}{\epsilon \omega} \ll 1, \quad \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2} = 1 + \frac{1}{2} \left(\frac{\sigma}{\epsilon \omega} \right)^2,$$

$$d \approx \frac{1}{\omega} \sqrt{\frac{2}{\epsilon \mu} \left[1 + \frac{1}{2} \left(\frac{\sigma}{\epsilon \omega} \right)^2 - 1 \right]}^{-1/2} = \frac{1}{\omega} \sqrt{\frac{2}{\epsilon \mu}} \sqrt{2} \left(\frac{\epsilon \omega}{\sigma} \right) = \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}}$$

good conductor:

$$\frac{\sigma}{\epsilon \omega} \gg 1, \quad d \approx \frac{1}{\omega} \sqrt{\frac{2}{\epsilon \mu} \left(\frac{\sigma}{\epsilon \omega} \right)^{-1}} = \sqrt{\frac{2}{\omega \mu \sigma}}, \quad k \simeq \kappa, \quad \lambda = \frac{2\pi}{k} = \frac{2\pi}{\kappa}, d = \frac{1}{k} = \frac{\lambda}{2\pi}$$

(b)

$$60\text{Hz: } \frac{\sigma}{\epsilon \omega} = \frac{22}{60 \cdot 80 \cdot \epsilon_0} = \frac{22}{60 \cdot 80 \cdot 8.85 \cdot 10^{-12}} = 5.2 \cdot 10^8 \gg 1 \dots \text{good conductor}$$

$$\rightarrow d \approx \sqrt{\frac{2}{\omega \mu \sigma}} = \sqrt{\frac{2}{2\pi \cdot 60 \cdot 4\pi \cdot 10^{-7} \cdot 22}} = 13.9(m)$$

$$60\text{GHz: } \frac{\sigma}{\varepsilon \omega} = 5.2 \cdot 10^8 \cdot \frac{1}{10^9} = 0.52 \dots \text{poor conductor}$$

$$\rightarrow d = \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}} = \frac{2}{22} \sqrt{\frac{80 \cdot 8.85 \cdot 10^{-12}}{4\pi \cdot 10^{-7}}} = 2.2 \cdot 10^{-3}(m)$$

5.

$$(a) \quad \omega_c = \frac{\pi c}{a}, f_c = \frac{c}{2a} = \frac{3 \cdot 10^8}{2 \cdot 0.072} = 2.08(\text{GHz})$$

(b)

$$B_z = B_0^+ \cos(\pi x/a) e^{i(kz - \omega t)} + B_0^- \cos(\pi x/a) e^{i(-kz - \omega t)} = \cos(\pi x/a) e^{-i\omega t} (B_0^+ e^{ikz} + B_0^- e^{-ikz})$$

Boundary:

$$z=0, B_z = 0, \Rightarrow B_0^+ + B_0^- = 0, \Rightarrow B_0^+ = -B_0^-, \Rightarrow B_z = 2iB_0^+ \cos(\pi x/a) e^{-i\omega t} \sin(kz)$$

$$z=d, B_z = 0, \Rightarrow \sin(kd) = 0, \Rightarrow kd = p\pi, p=1, 2, 3\dots$$

$$\Rightarrow B_z = 2iB_0^+ \cos(\pi x/a) \sin(p\pi z/d) e^{-i\omega t}$$

$$\begin{aligned} B_x &= \frac{-ika}{\pi} B_0^+ \sin(\pi x/a) e^{i(kz - \omega t)} + \frac{ika}{\pi} B_0^- \sin(\pi x/a) e^{i(-kz - \omega t)} = \frac{-ika}{\pi} \sin(\pi x/a) e^{-i\omega t} (B_0^+ e^{ikz} - B_0^- e^{-ikz}) \\ &= \frac{-ika}{\pi} \sin(\pi x/a) e^{-i\omega t} (B_0^+ e^{ikz} + B_0^+ e^{-ikz}) = \frac{-2ika}{\pi} B_0^+ \sin(\pi x/a) \cos(kz) e^{-i\omega t} \\ &= \frac{-2ia}{\pi} \left(\frac{p\pi}{d}\right) B_0^+ \sin(\pi x/a) \cos(p\pi z/d) e^{-i\omega t} \end{aligned}$$

$$\begin{aligned} E_y &= \frac{i\omega\mu a}{\pi} B_0^+ \sin(\pi x/a) e^{i(kz - \omega t)} + \frac{i\omega\mu a}{\pi} B_0^- \sin(\pi x/a) e^{i(-kz - \omega t)} = \frac{i\omega\mu a}{\pi} \sin(\pi x/a) e^{-i\omega t} (B_0^+ e^{ikz} + B_0^- e^{-ikz}) \\ &= \frac{i\omega\mu a}{\pi} \sin(\pi x/a) e^{-i\omega t} (B_0^+ e^{ikz} - B_0^+ e^{-ikz}) = \frac{-2\omega\mu a}{\pi} B_0^+ \sin(\pi x/a) \sin(kz) e^{-i\omega t} \\ &= \frac{-2\omega\mu a}{\pi} B_0^+ \sin(\pi x/a) \sin(p\pi z/d) e^{-i\omega t} \end{aligned}$$